FULL NOTES - CALCULUS

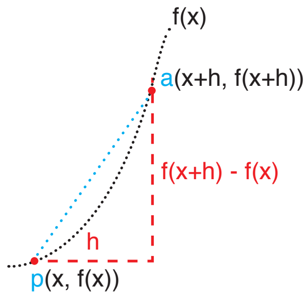
6.1. LIMITS / DERIVATIVES

➔ **LIMITS** ⇨ : As a sepuance continues forever, what point does it tend towards. This point is the limit.

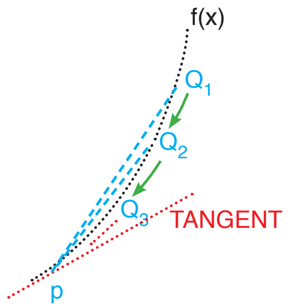
E.G.: The limit of 0.3 , 0,33 , 0,333, ... is 0,3 or 1/3 (It 'tend' to 1/3)

$\lim_{x\to\infty}\left(\frac{2x+3}{x-1}\right)$ is the, as you plug in x = 1,2,3,... and get the sequence : $\frac{5}{0}$, $\frac{7}{1}$, $\frac{9}{2}$, ... It tends to 2.

⇨ : This brings us to the main point, which invelves the limit of a sequence of gradients of lines. Explained below.



* Here, we have a function : y = f(x)
* We also have a point P(x, f(x)) & Q(x + h, f(x + h))
* We want to find the gradient of the line PQ.
* So we do $\frac{\text{change in y}}{\text{change in x}}$, which is $\frac{f(x+h) - f(x)}{h}$



* Now, we start looking at a series of points / lines
* That ' h ' valve is getting closer & closer to zero.
* We're going to look at what the gradients tend to.
* This is written $\lim_{h\to\infty}\frac{f(x+h) - f(x)}{h}$
* As h gets infinitely close to zero, it approximates the ' tangent '

* So finding this limit is equivalent to finding the gradient of the tangent.
* We call this : finding the derivative / f'(x) / d_y/d_x / differentiating

E.G.: Find the derivative of $f(x) = x^2$

$$\lim_{h\to\infty}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to\infty}\frac{x^2 + 2xh + h^2 - x^2}{h} = ...$$

$$= \lim_{h\to\infty}\frac{2xh + h^2}{h} = \lim_{h\to\infty} 2x + h = 2x$$

⟶ This means : if you take a point on $f(x) = x^2$, the gradient of the tangent at that point will be equal to 2x

6.1. LIMITS / DERIVATIVES

➔ **LIMITS** ⇨ : As a sepuance continues forever, what point does it tend towards. This point is the limit.

E.G.: The limit of 0.3 , 0,33 , 0,333, ... is 0,3 or 1/3 (It 'tend' to 1/3)

$\lim_{x \to \infty} \left(\frac{2x + 3}{x - 1} \right)$ is the, as you plug in x = 1,2,3,... and get the sequence : $\frac{5}{0}$, $\frac{7}{1}$, $\frac{9}{2}$, ... It tends to 2.

⇨ : This brings us to the main point, which invelves the limit of a sequence of gradients of lines. Explained below.

6.2. DERIVATIVE RULES

REMEMBER ⇨ : We will represent the derivature in 2 different ways.

1. Derivative of f(x) ⇨ f'(x)
2. Derivative of y = ⇨ dy/dx = we will use both

RULES ⇨

f(x)	f'(x)
x^n	nx^{n-1}
$a\text{-}x^n$	nax^{n-1}
sinx	cosx
cosx	-sinx
tanx	$1/\cos^3 x$
ex	ex
lnx	1/x

meaning x^3 ⇨ $3x^2$, x^6 $6x^5$, etc
so $2x^4$ ⇨ $8x^3$, $-7x^6$ ⇨ $-4x^5$, etc

sinx + cosx ⇨ 3cosx - sinx

$4e^2$ - 2lnx ⇨ $4e^2$ - 2/x

NOTE 1 ➪ If f(x) is the sum (or difference) of multiple component, then you can differentiate them separately,

EG . ➪ $6x^2$ - sinx ➪ $18x^2$ - cosx

NOTE 2 ➪ If a tem / component is multilied by a contant, the derivative keeps that constant, ınchaged.

EG . ➪ 4sinx ➪ 4cosx, $-9e^x$ $-9e^2$

FURTHER RULES

REMINDER ➪ : If we have $f(x) = x^2$ ℓ g(x) = 5x, then f(g(x)) = (5x)2 is a composite function. Below we learn hov to differentiate them.

CHANIN RULE ➪ d/dx f(g(x)) = f'(g(x)) g'(x)

EG 1 . ➪ $y = (3x^2 + 5x + 2)^7$ ➪ $dy/dx = 7(3x^2 + 5x + 2)^6$ (6x + 5)

EG 2 . ➪ f(x) = sin(3x) ➪ f'(x) = cos(3x) +3 = 3cos3x

EG 3 . ➪ $y = e^{x2-3x}$ ➪ $dy/dx = (2x - 3)\ e^{x2 - 3x}$

CALCULUS (continued)

We've seen examples of diffrentiating functions that are added together.

f(x) = v(x) + v(x) ➪ f'(x) = v'(x) + v'(x)

However, when v(x) & v(x) are multiplied together, it is not that simple :

PRODUCT RULE ➪ f(x) = u(x) v(x) f'(x) = u'(x) v(x) + u(x) v'(x)

[It is useful to write out u(x), v(x), u'(x) v'(x) clearly before you write an answer]

E.G 1 ⇨ $f(x) = x^2(2x-1)$ // $u(x) = x^2$, $v(x) = 2x-1$, $v'(x) = 2x$, $v'(x) = 2$

$f'(x) = 2x(2x - 1) \times x^2(2) = 4x^2 - 2x + 2x2 = 6x^2 - 2x$

E.G 2 ⇨ $f(x) = (3x^4)(\sin x)$ // $v(x) = 3x^4$, $v(x) = \sin x$, $v'(x) = 12x^3$, $v'(x) = \cos x$

$f'(x) = 12x^3 \sin x + 3x4 \cos x$

Next, we see a rule for when v(x) is divided by v(x) :

QUOTIENT RULE ⇨ $f(x) = \frac{u(x)}{v(x)}$ ⇨ $f'(x) = \frac{u'(x)v(x) - u(x)\,v'(x)}{(v(x))^2}$

E.G 1 ⇨ $y = \frac{1 + 3x}{2 - x}$ // $u = 1 + 3x$, $v = 2 - x$, $u' = 3$, $v' = -1$

$$\frac{dy}{dx} = \frac{3(2 - x) - (1 + 3x)(-1)}{(2 - x)^2} = \frac{6 - 3x + 1 + 3x}{(2 - x)^2} = \frac{7}{(2 - x)^2}$$

E.G 2 ⇨ $y = \frac{2\sqrt{x}}{2 - x}$ // $v = 2\sqrt{x}$, $v = 1 - x$, $v' = 1/\sqrt{x}$, $v' = -1$

$$\frac{dy}{dx} = \frac{(1 - x) - (1/\sqrt{x})(-1)}{(1 - x)^2} = \frac{(1 - x) + 2x}{\sqrt{x}\,(1 - x)^2} = \frac{1 + x}{\sqrt{x}\,(1 - x)^2}$$

There are various uses for finding the 'second derivative' of a function :

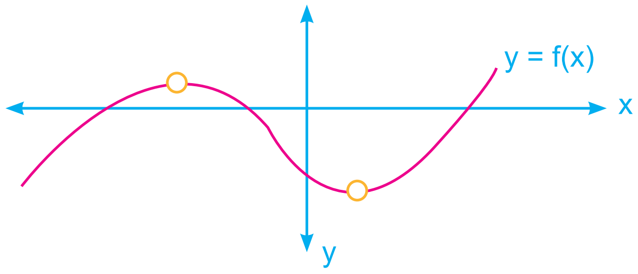
SECOND DERIVATIVE ⇨ : We simply diferentiate two times

NATOTION : $f''(x)$ / d^2y / dx^2

E.G. 1 ⇨ $f(x) = 7x^2 - x^3$ $f'(x) = 14x - 3x^2$ ⇨ $f''(x) = 14 - 6x$

6.3. VERTICES / OPTIMISATION

TERMONOLOGY ⇨ : Wertices, stationary point, turning points, maxima / minima are coloured red below



FINDING A TURNING POINT ⇨ : As the gradient of the line at the turning point is zero, we can use differentiation to find coordinates :

Solve f'(x) = 0 to find the x - coordinate.

E.G 1 ⇨ Find turning point of $f(x) = x^2$

ANS : $f'(x) = 3x^2$. So we must solve $3x^2 = 0$, $x = 0$. T.P. = (0,0)

E.G 2 ⇨ Find maximum of teh curve $y = 3x^2 = x^2$

ANS : $dy/dx = -6x + 12$ Solve $-6x + 12 = 0$, $x = 2$

$y = 3(2)^2 + 12(2)$. T.P. = (2,12)

$= -12 + 24 = 12$

INCREASINF DECREASING ⇨ : Find the valves of x for that $f'(x) > 0$ (positive) to, find where the curve is increasing. $f'(x) < 0$ for decreasing

E.G 1 ⇨ For what x valves is f'(x)) $2x^2 - 8$ increasing?

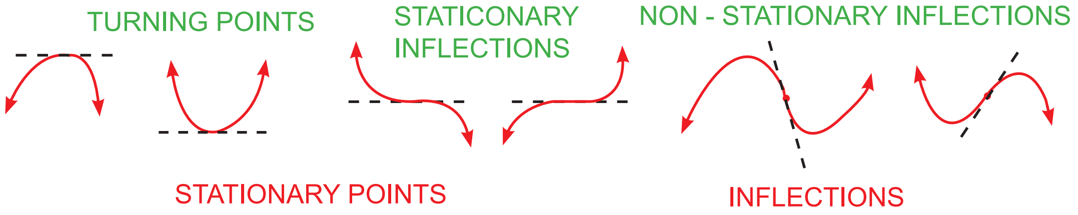
ANS : $f'(x) = 4x$. Solve $4x > 0$, $x > 0$

E.G 2 ⇨ For what x valves is $y = 1/3\ x^3 + 2x^2 - 5x + 6$ decreasing?

ANS : $dy/dx = x^2 + 4x - 5$. Solve $x^2 + 4x - 5 < 0$,

$(x + 5)(x - 1) < 0$, $-5 < x < 1$

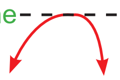
CLASSIFYING POINTS ⇨ OF INFLECTION



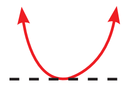
6.3.CONT

STATIONARY POINTS ⇨ [MAX, MIN OR INFLECTION] **:** We knox that we can find these using f'(x) = 0, but we need know what type of stationary point it is :

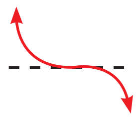
MAX ⇨ When f''(x) is negative after plugging in the x-coord of the turning point.



MIN ⇨ When f''(x) is positive fter plugging in the x-coord of the turning point.



POINT OF INFLECTION ⇨ When f''(x) = 0 after the sance process, the point is neither max. or min.



E.G 1 ⇨ Find and classfy stationary point of $f(x) = x^4 - 4x^3 + 5$

ANS : $f'(x) = 4x^3 - 12x^2 = 0$, $4x^3 (x - 3) = 0$, so $x = 0$ or 3

$f''(x) = 12x^2 - 24x$

$f''(0) = 12(0)^2 - 24(0) = 0$ ∴ at x = 0, we have a inflection point

$f''(3) = 12(3)^2 - 24(3) = 36$ ∴ at x = 0, we have a minimum (3, -22)

➔ **INFLECTION POINT** ⇨ **:** There are different from stationary point. They occcur when the rake of change of gradient is zero. In other word : when $f''(x) = 0$

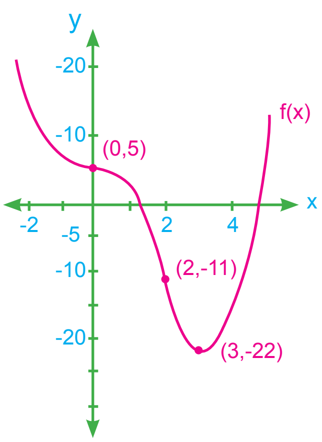
E.G 1 ⇨ Now find & classify inflection points of $f(x) = x^4 - 4x^3 + 5$

ANS : $f''(x) = 12x^3 - 24x$ [from above]

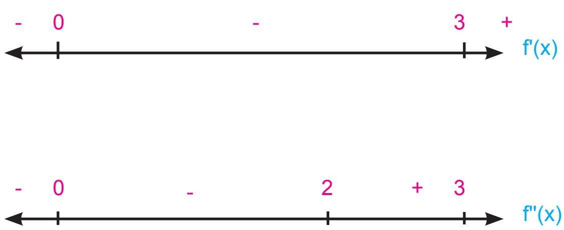
$12x^2 - 24x = 0$, $12x(x - 2) = 0$, so $x = 0$ or 2

$f''(0) = 0$ $f(0) = 5$ thus (0,5) is a stationary inflection

$f'(2) \neq f(2) = -11$, thus (2, -11) is a non stationry infelction :



We can represent this whit sign diagrams :



APPLICATIONS KINEMATICS

DISPLACEMENT ⇨ : The distance moved away from an origin, 0

Expressed as a function of time.

E.G. : $S(t) = t^2 + 2t - 3$ cm

VELOCITY ⇨ : Formal terminology for what you would call speed.

Impertantly, it is iqual to the rate of change of displacement.

E.G. : $S(t) = t^2 + 2t - 3$, then velocity [v, (t)] = $s'(t) = 2t + 2$ cm/s

So the velecity at 4 seconds would be $v(4) = 2(4) + 2 = 10$ cm/s

ACCELERATION ⇨ : The rate of change of velocity. This means it can be found with the derivative of velocity, or 2^{nd} derivative of displacement.

E.G. : If $v(t) = 2t + 2$, then $a(t) = v'(t) = 2$ cm/s^2

*** 'Speed' is the absolute valve of 'velocity'. [velocity can be negative]**

OPTIMISATION ➪ : This refers to any real - world problem that uses derivatives to help optimise a variable

EXAMPLES : Choosing the lengths of the sides of a box that maximises the volume.
* Finding which quantity of sales optimises profit

E.G 1 ➪ A rectangular dish is made by cutting the corners out of a 25x40 cm piece of tin, then folding up te metal.
Which size of comer maximises volume?

ANS : Find a function of x for volume :

➪ $V(x) = x(40 - 2x)(25 - 2x) = x\,(1000 - 80x - 50x + 4x^2)$

$= 1000x - 130x^2 + 4x^3 \ \ cm^3$

➪ Need to V'(x) = 0 to find maximum

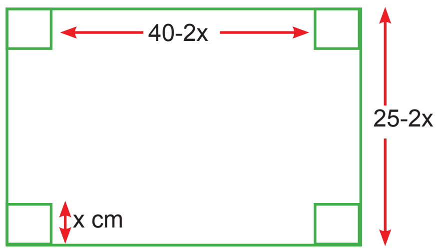
➪ $V'(x) = 1000 - 260x + 12x^2 = 0{+}4\ ,\ 250 - 65x + 3x^2 = 0$

$(3x - 50)(x - 5) = 0\ ,\ x = 5 \text{ or } 50/3$

➪ Check if V''(5) or V''(50/3) is negative to find maximum

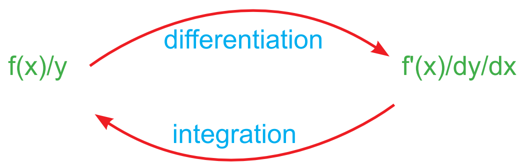
➪ $V''(x) = 24x - 260,\ V''(5) = 24(5) - 260 = -140$

Cut 5 cm squares out of the corner to maşimise volume



6.4.INTEGRATION

INTEGRATION ➪ : is essentially the pposite of differentiation



➪ As we say that $4x^3$ is the dirivative of x^4, we can then say that x^4 is the 'integral' or 'antiderivative' of $4x^3$

NOTATION ⇨ : The integral of f(x) i written $\int f(x)\, d.x$

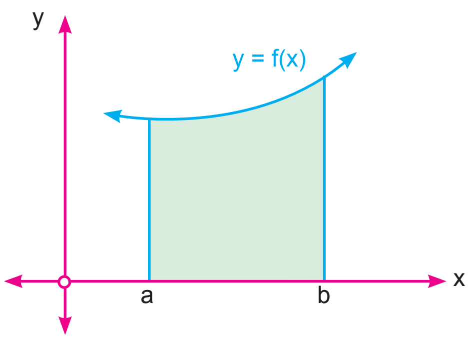
E.G. : $\int 4x^3\, dx = x^4 + c$

↳ We add a potential constant, as these disappear when differentiating.

AREA UNDER CURVES ⇨ : We can use integration to help find the area under a curve (see diagram)

⇨ To find this shaded area, we can use

$\int_a^b f(x)\, dx = F(b) - F(a) = [F(x)]_a^b$



E.G.1 : $\int_2^3 4x^3\, dx = F(3) - F(2) = (3)^4 - (2)^4 = 65$

↳ $(F(x) = x^4)$

↳ **RULES :**

$\int_a^b c\, dx = c(b - a)$

$\int_b^a f(x)\, dx = \int_a^b f(x)\, dx$

$\int_a^b f(x)\, dx = \int_b^c f(x)\, dx \quad \int_a^c f(x)\, dx$

$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$

↳ **SPECIFIC CASES** ⇨ :

Function	Integral
k(constant)	kx+c
x^n	$x^{n+1}/n+1$ +c
e^x	e^x + c
1/x	In|x| + c
cosx	sinx + c
sinx	-cosx +

E.G.1 : $\int 5\,dx = 5x + c$

E.G.2 : $\int x^6\,dx = \frac{x^2}{7} + c$

E.G.3 : $\int (3x + 2/x)^2\,dx$ [No product rule]

$= \int (9x^2 + 12 + 4x^{-2})dx$ [Expand first]

$= \int \frac{9x^3}{3} + 12x + \frac{4x^{-1}}{-1} + c = 3x3 + 12x - 4/x + c$

↳ **FURTHER CASES ⇨ :**

Function	Integral
e^{nx+b}	$1/a\ e^{nx+b}$
$(ax + b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
$\frac{1}{ax + b}$	$\frac{1}{a}\ \ln\|ax + b\| + c$
$\cos(ax + b)$	$\frac{1}{a}\sin(ax + b) + c$
$\sin(ax + b)$	$\frac{-1}{a}\cos(ax + b) + c$

E.G.1 : $\frac{1}{2} = \frac{(2x + 3)^5}{5} + c = \frac{(2x + 3)^5}{5} + c$

E.G.2 : $\int \frac{4}{1 - 2x}\,dx = 4\int \frac{14}{1 - 2x}dx = 4x\left(\frac{1}{2}\right) x \ln|1 - 2x| \times c = 2\ln|1 - 2x| + c$

PARTICULAR VALVES ⇨ : Instead of leaving your answer with a ' + c ' at the end evry time, sometimes you will be given enough information to plug in valves, and solve for c.

E.G.1 : Find f(x) given that $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$

↳ SOL : $f(x) = \int (x^3 - 2x^2 + 3)\,dx = \frac{x^2}{4} - \frac{2x^2}{3} + 3x + c$

$f(0) = \frac{0}{4} - \frac{0}{3} + 3(0) + c = 2,\ \ c = 2\ \ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + 2$

SUBTITUTION ➪ **:** This a method of integrating used when we have an expression multiplied by its own derivative :

$$\int f(x) \frac{dy}{dx} dx = \int f(x)\, dx$$

E.G. 1 : $\int (x^2 + 3x)^4 (2x + 3)\, dx$ - Notice that 2x + 3 is the derivative of $x^n + 3x$

$= \int (u)^4 \frac{du}{dx} dx$ - Substitute $u = x^2 + 3x$

$= \int (u)^4 du \frac{u^5}{5} + c$ - Integrate w . r . t . u

$= \frac{1}{5}(x^2 + 3x)^5 + c$ - Substitute back

FINDING AREA (Definite Integrals) ➪ **:**

E.G. 1 : $\int_1^3 (x^2 + 2)\, dx$

$$= \left[\frac{x^3}{3} + 2x\right]_1^3 = \left(\frac{27}{3} + 2(3)\right) - \left(\frac{1}{3} + 2(1)\right)$$

$$= (9 + 6) - \left(\frac{1}{3} + 2\right) = 12\frac{2}{3}$$

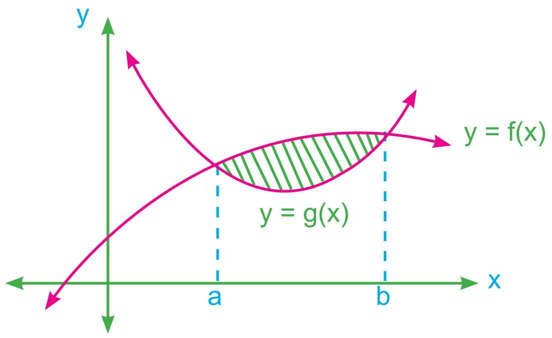
E.G. 2 : $\int_0^{\pi/3} \sin x\, dx$

$= [-\cos x]_0^{\pi/3} = (-\cos \pi/3) - (-\cos 0)$

$= 1/2 + 1 = 1/2$

6.5. INT. APPLICATIONS

AREA BETWEEN TWO CURVES ➪ **:** A natural extension of the method for fuding area under a curve.

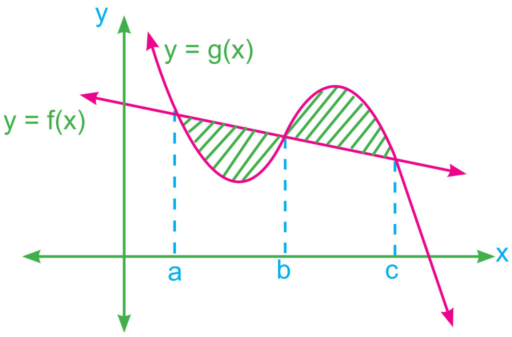


$$\text{Area} = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b f(x) - g(x)\, dx$$

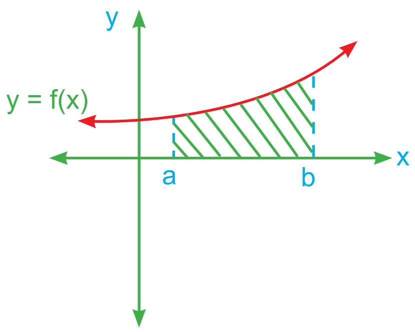
SPECIAL CASE ➪ : This is not a simple, because g(x) is above f(x) for a section of the area

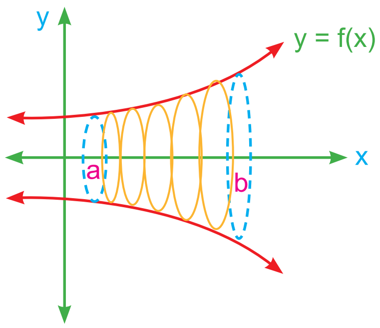
This would give a ' negative ' area



So Area $= {}_a\int^b f(x) - g(x)\,dx \; - \; {}_b\int^c f(x) - g(x)\,dx$

SOLIDS OF REVOLUTION ➪ :





We can use a combination of integration and the formula for volume of a cylinder to find the volume for the solid created by rotating f(x) 360° around the x-axis

E.G. 1 : Find the volume of he solid formed when the graph of the function $y = x^2$ for $0 \le x \le 5$ is revoved 2π about the x-axis

Volume $= \pi \; {}_a\int^b y^2\,dx = \pi \; {}_0\int^5 (x^2)^2\,dx = \pi \; {}_0\int^5 x^4\,dx = \pi \left[\frac{x^5}{5}\right]_0^5$

$= \pi \left(\frac{3125}{5} - \frac{0}{5}\right) = 625\,\pi \text{ units}^3$

6.6. KINEMATICS

> NOTE 1 ⇨ Much of the kinematics content has already been conveered in
> 6.3 - APPLICATIONS OF DIFFERENTIATION

Just as we have our basic kinematics differentiation rules :

$s(t)$, $v(t) = ds/dt$, $a(t) = dv/dt = d^2s / dt^2$

We have the carrecfonding rules for intergration :

$s(t)$, $v(t) = \int a(t)\, dt$, $s(t) = \int v(t)\, dt$

From this, we can get a formula for distance travelled :

For a velocity - time function v(t)

Where v(t) = 0 for the internal $t_1 \leq t \leq t_2$:

Dist travelled = $\int_{t_1}^{t_2} v(t)\, dt$

By extension, distance travelled is the area under a velecity -time graph

(if velecity stays in the some direction)

E.G. 1 : A particle has velocity function v(t) = 1 . 2t cms^{-1} as it moves in a straight line. The particle starts 2m to the right of 0.

a) Write a formula for displacement s(t) :

$\int v(t)\, dt =$ (1 - 2t) dt $s(t) = (t - t^2 + c)$ cm c = 2, $s(t) = (t - t^2 + 2)$ cm

b) Find the total distance travelled in the first second of motion :

After 1/2 sec, velocity becomes negative. So we must suldract the integral from 1/2 to 1 :

Disk travelled = $\int_0^{1/2} v(t)\, dt - \int_{1/2}^{1} v(t)\, dt = [t - t^2 + 2]_0^{1/2} - [t - t^2 + 2]_{1/2}^{0}$

= ((1/2 - 1/4 + 2) - (2)) - ((1 - 1 + 2) - (1/2 - 1/4 + 2)) = (1/4) - (-1/4) = 1/2 cm

c) Find dislacement at the end of one second :

Use displacement = final position - original position

= s(1) - s(0)

= (1 - 1 + 2) - (0 - 0 + 2) = 0 cm