TOPİC I

ALGEBRA FULL NOTES

1.1. SEQUENCES SERIES

➔ **BASICS:** There are two main types of sequence:

- **ARİTMETHIC:** add by d each term. E. G: 1, 4, 7, 10, ...
- **GEOMETRIC:** multiply by r each term. E. G: 2, 6, 18, 54, ...

➔ **n^{th} TERM:** We can an aritmethic sequence using a general term formula (or n^{th} term formula)

ARİTMETHIC:

$$u_n = u_1 + (n-1)d$$

term number | 1^{st} term | common difference

E.G.: 20^{th} term of –3, –1, 1, 3, 5, ... ?

ANS: $u_1 = -3$, $n = 20$, $d = 2$

So $u_{20} = -3 + (20 - 1) \cdot 2 = -3 + (19 \cdot 2) = 35$

All formulas in pink are given in the formula book.

GEOMETRIC:

$$u_n = u_1 \cdot r^{n-1}$$

term number | 1^{st} term | $(\text{common ratio})^{n-1}$ →

You can find this by taking any term and dividing it by the previous term

E.G: 6^{th} term of 2, 10, 50 ... ?

ANS: $u_1 = 2$, $n = 6$, $r = \frac{10}{2}$ (or $\frac{50}{10}$) $= 5$

So $u_6 = 2 \cdot 5^{6-1} = 2 \cdot 5^5 = 2 \cdot 3125 = 6250$

E.G: 7^{th} term of 12, –6, 3, $-\frac{3}{2}$, ... ? *[common ratio can be a fraction too]*

ANS: $u_1 = 12$, $n = 7$, $r = -\frac{6}{12} = -\frac{1}{2} = 5$

So $u_7 = 12 \cdot (-\frac{1}{2})^{7-1} = 12 \cdot \frac{1}{64} = \frac{12}{64} = \frac{3}{16}$

➔ SUM / SERIES

- **S_n:** S_n means $u_1 + u_2 + u_3 + \dots + u_n$
- **'Sigma' NOTATİON:** $\sum_{k=1}^{n} u_k$ → $\boxed{\text{the sum of all } u_k\text{'s from 1 on n } [u_1 + u_2 + \dots + u_k]}$

ARİTMETHIC SUM: $S_n = \frac{n}{2}(u_1 + u_n)$ or $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

E.G.: Find sum of 3 + 7 + 11 + 15 + ... to 20 terms.

ANS: $n = 20,\ u_1 = 3,\ d = 4$

So, $u_{20} = \frac{20}{2}((2 \cdot 3) + (20 - 1) \cdot 4) = 10 \cdot (6 + 76) = 820$ → $\boxed{\text{using } 2^{nd} \text{ formula}}$

E.G.: Find sum of 5 + 8 + 11 + ... + 101

ANS: $n = 32,\ u_1 = 5,\ u_n = 101$

So, $u_{32} = \frac{32}{2}(5 + 101) = 16 \cdot (106) = 1696$ → $\boxed{\text{using } 1^{st} \text{ formula}}$

GEOMETRIC SUM: $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$ or $S_n = \dfrac{u_1(1 - r^n)}{r - 1}$

E.G.: Find sum of 2 + 6 + 18 + 54 + ... to 12 terms.

ANS: $u_1 = 2,\ r = 3,\ n = 12$

So, $u_{12} = \dfrac{2(3^{12} - 1)}{3 - 1} = 531.440$ → $\boxed{\text{using } 1^{st} \text{ formula}}$

INFINITE GEOMETRIC SUM: $S_n = \dfrac{u_1}{1 - r}$ when $-1 < r < 1$

E.G.: Show $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 2$

ANS: Could be written $\sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1}$

$r = \frac{1}{2},\ u_1 = 1$ so $\dfrac{1}{1 - \frac{1}{2}} = \dfrac{1}{\frac{1}{2}} = 2$

1.2. LOGS / EXPONENTIALS

➔ **EXPONENTS:** The following rules are **NOT** in the formula booklet.

- $a^m \cdot a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m+n}, \quad a \neq 0$
- $(a^m)^n = a^{m \cdot n}$
- $(ab)^n = a^n \cdot b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^0 = 1, \quad a \neq 0$
- $a^{-n} = \frac{1}{a^n}$

E.G.1: $5^4 \cdot 5^7 = 5^{11}$

E.G.2: $\frac{k^2}{k^3} = k^5$

E.G.3: $8 \cdot 2^t = 2^3 \cdot 2^t = 2^{t+3}$

E.G.4: $\frac{9}{27^t} = \frac{3^2}{3^{3t}} = 3^{2-3t}$

E.G.5: $\left(\frac{4}{3}\right)^{-2} = \frac{4^{-2}}{3^{-2}} = \frac{\frac{1}{16}}{\frac{1}{9}} = \frac{9}{16}$

➔ **RATIONAL EXPONENTS:** When powers are written as fractions.

- $a^{\frac{1}{2}} = \sqrt{a}$
- $a^{\frac{1}{3}} = \sqrt[3]{a}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

→ Also not in formula booklet.

E.G.1: $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

E.G.2: $27^{-\frac{2}{3}} = (3^3)^{-\frac{2}{3}} = 3^{-2} = \frac{1}{9}$

E.G.3: $16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$

➔ **EXPANSION / FACTORİNG:** Combine the rules below, and ones above.

- $a(b + c) = ab + ac$
- $(a + b)(c + d) = ac + ad + bc + bd$
- $(a + b)(a - b) = a^2 - b^2$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$

E.G.1: $2^x \cdot (2^x + 1) = 2^x \cdot 2^x + 2^x = 2^{2x} + 2^x = 4^x + 2^x$

E.G.2: $(3^x + 2)(3^x + 5) = 3^{2x} + 7 \cdot 3^x + 10$

E.G.3: $3^{n+2} + 3^n = 3^n(3^n + 1)$

E.G.4: $4^x - 25 = (2^x + 5)(2^x - 5)$

E.G.5: $\frac{20^n}{4^n} = \frac{5^n \cdot 4^n}{4^n} = 5^n$

➔ **EXPONANTIAL EQUATIONS:** Use $a^x = a^k$, then $x = k$

E.G.1: $2^x = 8 = 2^x = 2^3, \ x = 3$

E.G.2: $7^{x+1} = 343 = 7^{x+1} = 7^3, \ x + 1 = 3, \ x = 2$

E.G.3: $4^{2x+1} = 8^{1-x}, \ (2^2)^{2x+1} = (2^3)^{1-x}, \ 2^{4x+2} = 2^{3-3x}, \ 4x + 2 = 3 - 3x, \ 7x = 1, \ x = \frac{1}{7}$

E.G.4: $3 \cdot 2^{x+1} = 24, \ 3 \cdot 2^{x+1} = 3 \cdot 2^3, \ x + 1 = 3, \ x = 2$

➔ LOGARITHMS

GENERAL RULE → $\boxed{\text{If } b = a^x \text{ then } x = \text{Log}_a b}$ in formula booklet

E.G.: $3^x = 81$, $x = \text{Log}_3 81 = 4$

FIRST RULES → $x = \text{Log}_a a^x$ // $x = a^{\text{Log}_a x}$

EG - 1 : $\text{Log}_5 0.2 = \text{Log}_5\left(\frac{1}{5}\right) = \text{Log}_5 5^{-1} = -1$

EG - 2 : $\text{Log}_2\left(\frac{1}{\sqrt{2}}\right) = \text{Log}_2 2^{\frac{1}{2}} = -\frac{1}{9}$

➔ LAWS

In form book

$\text{Log}_c A + \text{Log}_c B = \text{Log}_c(AB)$

$\text{Log}_c A - \text{Log}_c B = \text{Log}_c\left(\frac{A}{B}\right)$

$n\text{Log}_c A = \text{Log}_c(A^n)$

E.G.1: $\text{Log}5 + \text{Log}3 = \text{Log}(3\text{x}5) = \text{Log}15$

E.G.2: $2\text{Log}7 - 3\text{Log}2 = \text{Log}47 - \text{Log}8 = \text{Log}\,\frac{47}{8}$

E.G.3: $2\text{Log}3 + 3 = \text{Log}(3^2) + \text{Log}(10^3) = \text{Log}9 + \text{Log}1000 = \text{Log}9000$

E.G.4: $\text{Log}A = \text{Log}b + 2\text{Log}c$, $\text{Log}A = \text{Log}B + \text{Log}c^2 = \text{Log}(bc^2)$: $A = bc^2$

NOTE : ' lnx' means $\text{Log}_e x$ (e is the 'natural exponential = 2.718..)

E.G.5: $\ln e^2 = \text{Log}_e e^2 = 2$ //

E.G.6: $e^{2\ln 3} = e^{\ln 9} = e^{\text{Log}_e 9} = 9$

➔ 'TALKING LOGS" (of each side)

- We use the fact that if $x = y$, then $\text{Log}x = \text{Log}y$

EG - 1 : $2^x = 30$, $\text{Log}2^x = \text{Log}30$, $x\text{Log}2 = \text{Log}30$, $x = \frac{\text{Log}30}{\text{Log}2}$

➔ CHANGE OF BASE

RULES → $\text{Log}_b a = \frac{\text{Log}_c a}{\text{Log}_c b}$

E.G.1: $\text{Log}_2 9 = \frac{\text{Ln}9}{\text{Ln}2} \approx 3.17$

OR : $\frac{\text{Log}_{10} 9}{\text{Log}_2 9} \approx 3.17$

➔ REAL WORLD Q'S

Q : Investment of \$ 5000, 5.2% p.a. interest, how long until \$20000?

SoL : Segvence of $U_n = 5000 \cdot (1.052)^{n-1}$, use $20000 = 5000 \cdot (1.052)^{n-1}$

so $(1.052)^{n-1} = 4$, $\log(1052)^{n-1} = \text{Log}4$, $n - 1 = \frac{\text{Log}4}{\text{Log}1.052} \approx 27.3$ yrs

➔ BINOMIAL EXPANSION

INVESTIGATION

Expand the following →

$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$

$(a+b)^3 = (a+b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3$

MEANING

Binomial expansion refers to the study and analysis of patterns that are created when $(a+b)^n$ is expanded, with any n valve

COEFFICIENTS

These are the constants that multiply each term of the expansion, marked in red in the above examples. We will look at four methads for finding them

1 Manual expansion

This is what we did above, but this will start to become a difficult task when we get to $(a+b)^5$, and larger n's

2 Pascal's triangle

This triangle is created by simply adding the two numbers immediately above :

	1
n = 1	1 1
n = 2	1 2 1
n = 3	1 3 3 1
n = 4	1 4 6 4 1
n = 5	1 5 10 10 5 1
n = 6	1 6 15 20 15 6 1

EG - 1 : Expand $(a+b)^5$: We will use the n = 5 row ⇒

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

EG - 2 : Find coefficient for the x^2 term in expansion of $(x+3)^4$

Here, a = x b = 3 So we use the $5a^2b^2$ term : $6x^2(3^2) = 54x^2$

3 Factorial formula

We can use the notation $\binom{n}{r}$ or nCr to represent these cofficients, where n is the power, and r is the position in the expansion (Starting at 0)

Then there is a formula : $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

EG - 1 : $\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{120}{6.2} = 10$ ⇒ You can check this by looking at the relevant section of Pascal's triangle

4 GDC use

If this is a paper 2 question, you work out this nCr velve by simply finding the nCr button on your GDC

➔ IB QUESTION SOLVING :

You will most likely be asked to find the coefficient of a specific term in an expanion. The coefficient will usually also be multiplied by the b^r valve as well

NOTE : There are (n +1) terms in the expansion of (a + b)n

NOTE : The 'Binomial Theorem' sums uf the whole expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \ldots\ldots + \binom{n}{r} a^{n-r} b^r + \ldots\ldots b^n$$

EG - 1 : Write the first three terms of $(1 + 2x)^{11}$

↳ **SOL :** $1^{11} + \binom{11}{1} 1^{10} (2x)^1 + \binom{11}{2} 1^9 (2x)^2 = 1 + (11)(1)(2x) + (55)(1)(4x^2)$

$= 1 + 22x + 220x^2$

EG - 2 : Write the fouurth term of $(2x + 5)^{15}$:

↳ **SOL :** $\binom{15}{3} (2x)^{12} (5)^3 = (455)(4096x^{12})(125) = 232960000x^{12}$

EG - 3 (IB) : Consider the expansion $(x + 3)^{10}$

a) Write down the number of terms in this expansion

↳ **SOL :** 10 + 1 = 11

b) Find the term containing x^3 :

↳ **SOL :** n = 10 , r = 7 $\binom{10}{7} x^3\ 3^7 = (120)\ x^3\ (2187) = 262440\ x^3$

EG - 4 (IB) : The 5^{th} term in the expansion of $(a + b)^n$ is given by $\binom{10}{4} p^6 (2g)^4$:

a) Write down the valve of ın : 10

b) Write down a & b : p & 2g

c) Write down an expression for the 6th term :

↳ **SOL :** $\binom{10}{5} p^5 (2g)^5 = (32g^5) = 8064\ p^5\ g^5$