TOPİC I

ALGEBRA — FULL NOTES

1.1. SEQUENCES SERIES

➔ **BASICS:** There are two main types of sequence:

- **ARİTMETHIC:** add by d each term. E. G: 1, 4, 7, 10, ...
- **GEOMETRIC:** multiply by r each term. E. G: 2, 6, 18, 54, ...

➔ **n^{th} TERM:** We can an aritmethic sequence using a general term formula (or n^{th} term formula)

ARİTMETHIC:

$$u_n = u_1 + (n-1)d$$

u_n ← term number; u_1 ← 1^{st} term; d ← common difference

E.G.: 20^{th} term of −3, −1, 1, 3, 5, ... ?

ANS: $u_1 = -3$, $n = 20$, $d = 2$

So $u_{20} = -3 + (20-1) \cdot 2 = -3 + (19 \cdot 2) = 35$

All formulas in pink are given in the formula book.

GEOMETRIC:

$$u_n = u_1 \cdot r^{n-1}$$

u_n ← term number; u_1 ← 1^{st} term; r^{n-1} ← (common ratio)$^{n-1}$ → You can find this by taking any term and dividing it by the previous term

E.G: 6^{th} term of 2, 10, 50 ... ?

ANS: $u_1 = 2$, $n = 6$, $r = \frac{10}{2}$ (or $\frac{50}{10}$) $= 5$

So $u_6 = 2 \cdot 5^{6-1} = 2 \cdot 5^5 = 2 \cdot 3125 = 6250$

E.G: 7^{th} term of 12, −6, 3, $-\frac{3}{2}$, ... ? *[common ratio can be a fraction too]*

ANS: $u_1 = 12$, $n = 7$, $r = -\frac{6}{12} = -\frac{1}{2} = 5$

So $u_7 = 12 \cdot (-\frac{1}{2})^{7-1} = 12 \cdot \frac{1}{64} = \frac{12}{64} = \frac{3}{16}$

→ **SUM / SERIES**

- **S_n**: S_n means $u_1 + u_2 + u_3 + ... + u_n$
- **'Sigma' NOTATİON:** $\sum_{k=1}^{n} u_k$ → the sum of all u_k's from 1 on n $[u_1 + u_2 + ... + u_k]$

ARİTMETHIC SUM: $S_n = \frac{n}{2}(u_1 + u_n)$ or $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

E.G.: Find sum of 3 + 7 + 11 + 15 + ... to 20 terms.

ANS: $n = 20$, $u_1 = 3$, $d = 4$

So, $u_{20} = \frac{20}{2}((2 \cdot 3) + (20 - 1) \cdot 4) = 10 \cdot (6 + 76) = 820$ → using 2^{nd} formula

E.G.: Find sum of 5 + 8 + 11 + ... + 101

ANS: $n = 32$, $u_1 = 5$, $u_n = 101$

So, $u_{32} = \frac{32}{2}(5 + 101) = 16 \cdot (106) = 1696$ → using 1^{st} formula

GEOMETRIC SUM: $S_n = \frac{u_1(r^n - 1)}{r - 1}$ or $S_n = \frac{u_1(1 - r^n)}{r - 1}$

E.G.: Find sum of 2 + 6 + 18 + 54 + ... to 12 terms.

ANS: $u_1 = 2$, $r = 3$, $n = 12$

So, $u_{12} = \frac{2(3^{12} - 1)}{3 - 1} = 531.440$ → using 1^{st} formula

INFINITE GEOMETRIC SUM: $S_n = \frac{u_1}{1 - r}$ when $-1 < r < 1$

E.G.: Show $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... = 2$

ANS: Could be written $\sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1}$

$r = \frac{1}{2}$, $u_1 = 1$ so $\frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

1.2. LOGS / EXPONENTIALS

→ **EXPONENTS:** The following rules are **NOT** in the formula booklet.

- $a^m \cdot a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m+n}, \quad a \neq 0$
- $(a^m)^n = a^{m \cdot n}$
- $(ab)^n = a^n \cdot b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^0 = 1, \quad a \neq 0$
- $a^{-n} = \frac{1}{a^n}$

E.G.1: $5^4 \cdot 5^7 = 5^{11}$

E.G.2: $\frac{k^2}{k^3} = k^5$

E.G.3: $8 \cdot 2^t = 2^3 \cdot 2^t = 2^{t+3}$

E.G.4: $\frac{9}{27^t} = \frac{3^2}{3^{3t}} = 3^{2-3t}$

E.G.5: $\left(\frac{4}{3}\right)^{-2} = \frac{4^{-2}}{3^{-2}} = \frac{\frac{1}{16}}{\frac{1}{9}} = \frac{9}{16}$

→ **RATIONAL EXPONENTS:** When powers are written as fractions.

- $a^{\frac{1}{2}} = \sqrt{a}$
- $a^{\frac{1}{3}} = \sqrt[3]{a}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

→ Also not in formula booklet.

E.G.1: $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

E.G.2: $27^{-\frac{2}{3}} = (3^3)^{-\frac{2}{3}} = 3^{-2} = \frac{1}{9}$

E.G.3: $16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$

→ **EXPANSION / FACTORİNG:** Combine the rules below, and ones above.

- $a(b + c) = ab + ac$
- $(a + b)(c + d) = ac + ad + bc + bd$
- $(a + b)(a - b) = a^2 - b^2$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$

E.G.1: $2^x \cdot (2^x + 1) = 2^x \cdot 2^x + 2^x = 2^{2x} + 2^x = 4^x + 2^x$

E.G.2: $(3^x + 2)(3^x + 5) = 3^{2x} + 7 \cdot 3^x + 10$

E.G.3: $3^{n+2} + 3^n = 3^n(3^n + 1)$

E.G.4: $4^x - 25 = (2^x + 5)(2^x - 5)$

E.G.5: $\frac{20^n}{4^n} = \frac{5^n \cdot 4^n}{4^n} = 5^n$

→ **EXPONANTIAL EQUATIONS:** Use $a^x = a^k$, then $x = k$

E.G.1: $2^x = 8 = 2^x = 2^3$, $x = 3$

E.G.2: $7^{x+1} = 343 = 7^{x+1} = 7^3$, $x + 1 = 3$, $x = 2$

E.G.3: $4^{2x+1} = 8^{1-x}$, $(2^2)^{2x+1} = (2^3)^{1-x}$, $2^{4x+2} = 2^{3-3x}$, $4x + 2 = 3 - 3x$, $7x = 1$, $x = \frac{1}{7}$

E.G.4: $3 \cdot 2^{x+1} = 24$, $3 \cdot 2^{x+1} = 3 \cdot 2^3$, $x + 1 = 3$, $x = 2$

➔ LOGARITHMS

GENERAL RULE → If $b = a^x$ then $x = \text{Log}_a b$ → in formula booklet

E.G.: $3^x = 81$, $x = \text{Log}_3 81 = 4$

FIRST RULES → $x = \text{Log}_a a^x$ // $x = a^{\text{Log}_a x}$

EG - 1 : $\text{Log}_5 0.2 = \text{Log}_5\left(\frac{1}{5}\right) = \text{Log}_5 5^{-1} = -1$

EG - 2 : $\text{Log}_2\left(\frac{1}{\sqrt{2}}\right) = \text{Log}_2 2^{\frac{1}{2}} = -\frac{1}{9}$

➔ LAWS

In form book

$\text{Log}_c A + \text{Log}_c B = \text{Log}_c(AB)$

$\text{Log}_c A - \text{Log}_c B = \text{Log}_c\left(\frac{A}{B}\right)$

$n\text{Log}_c A = \text{Log}_c(A^n)$

E.G.1: Log5 + Log3 = Log(3x5) = Log15

E.G.2: 2Log7 - 3Log2 = Log47 - Log8 = $\text{Log}\,\frac{47}{8}$

E.G.3: 2Log3 + 3 = $\text{Log}(3^2) + \text{Log}(10^3)$ = Log9 + Log1000 = Log9000

E.G.4: LogA = Logb + 2Logc, LogA = LogB + $\text{Logc}^2 = \text{Log}(bc^2)$: $A = bc^2$

NOTE : ' lnx' means $\text{Log}_e x$ (e is the 'natural exponential = 2.718..)

E.G.5: $\ln e^2 = \text{Log}_e e^2 = 2$ //

E.G.6: $e^{2\ln 3} = e^{\ln 9} = e^{\text{Log}_e 9} = 9$

➔ 'TALKING LOGS" (of each side)

- We use the fact that if x = y, then Logx = Logy

EG - 1 : $2^x = 30$, $\text{Log}2^x = \text{Log}30$, xLog2 = Log30 , $x = \frac{\text{Log}30}{\text{Log}2}$

➔ CHANGE OF BASE

RULES → $\text{Logba} = \frac{\text{Log}_c a}{\text{Log}_c b}$

E.G.1: $\text{Log}_2 9 = \frac{\text{Ln}9}{\text{Ln}2} \approx 3.17$

OR : $\frac{\text{Log}_{10} 9}{\text{Log}_2 9} \approx 3.17$

➔ REAL WORLD Q'S

Q : Investment of $ 5000, 5.2% p.a. interest, how long until $20000?

SoL : Segvence of Un = 5000 . $(1.052)^{n-1}$, use 20000 = 5000 . $(1.052)^{n-1}$

so$(1.052)^{n-1} = 4$, $\log(1052)^{n-1}$ = Log4, $n - 1 = \frac{\text{Log}4}{\text{Log}1.052} \approx 27.3$ yrs

$a \times 10^k$ FORM

Numbers in the form: $a \times 10^k$, such as 4.7×10^5 instead of 470,000 , may be called scientific notation or standard form.

The number 470,000 could theoretically be written: $4.7 \times 10^5, 47 \times 10^4, 0.47 \times 10^6$, and so on.

However, to simplify things, we say that ' a ' & ' k ' must have the following:

- $1 \leqslant |a| < 10$
- $k \in \mathbb{Z}$

meaning: ' a ' is between 1 and 10 and ' k ' is an integer.

SIGNIFICANT FIGURES

➡ Zeros before the first non-zero number are not significant (leading zeros)

➡ All other zeros and other digits are 'significant'

DECIMAL PLACES

➡ Rounding to 1dec. place is the same as rounding to the nearest tenth.

➡ Rounding to 2 dip. is like rounding to the nearest hundreth, etc.

When rounding to n sig. figs, if the $(n+1)^{\text{th}}$ figure is < 5, keep the n^{th} figure the same. If $\geqslant 5$, then increase the n^{th} figure by one.

E.G.1 ➡ Round 0.00726 to 2sig. figs.

➡ Go to ' $2'$, check '6', raise '2' to '$3'$: 0.0073

E.G.2 ➡ Round 43.8037 to 2 dec. places:

➡ '0'is 2^{nd} dep. '$,3' < 5$, so keep '$0'$: 43.80

UPPER& LOWER BOUND

This refers to the process of saying how low or high a number could have been before rounding. For example, 3.1 (to 2 sig. figs.) could have been as low as 3.05 , and as high (but not including).

3.15 ,ie.: $3.05 \leqslant x < 3.15.$

EG.3. If $x = 7860$ to 3 sig. figs., give an upper & lower bound for x :

➡ $7855 \leq x < 7865$

E.G.4. ➡ A circle has radius of 3.1 cm, to 1 dep., give an upper bound for the area:

➡ with $r = 3.1$, area $= 30.19\ \text{cm}^2$. U. Bound of radius $= 3.15$ cm, max.

area $= 3.15^2 \times \pi = 9.9225\pi = 31.17 \ldots$

PERCENTAGE ERROR

As a percentage of the actual answer, how far off is your estimate/guess/rounded answer? This is what is called the percentage error.

$$ERROR = \left| \frac{V_{[\text{actual}]} - V_{[\text{estimated}]}}{V_{[\text{actual}]}} \right| \times 100\%$$

➡ **E.G.** Find the maximum error for area, if you used 3.1 cm as your radius in E.G.4:

➡ With a rounded radius of 3.1 cm, area would be $30.19\ \text{cm}^2$. Given the U.B. of 3.15 , the maximum area could have been $31.17\ \text{cm}^2$. So, $\left|\frac{31.17-30.19}{31.17}\right| \times 100\% = 3.14\%$

EG. ➡ Est $= 3000$, actual $= 3203.5$, find error:

$$= \left|\frac{3203.5 - 3000}{3203.5}\right| \times 100\% = 6.35\%$$

➡ You should round to 3 significant figures in exams unless told otherwise.

COMPOUND INTEREST

- ✓ Interest represents a modest percentage that a bank or company adds to an initial investment.
- ✓ Additionally, it can denote an amount paid on a loan or debt. However, in the context of IB compound interest questions, the focus is always on interest earned from investments.

Compound interest stands for the interest calculated on **both the initial investment** and any **previously accrued interest.**

- ✓ Simple interest only applies to the original investment

The interest paid each time will increase as it is a percentage of a higher number.

Compound interest will be paid in instalments in a given timeframe.

- ✓ The interest rate, *r*, will be per annum (per year). This could be written *r* % p.a.
- ✓ Keep an eye out for terms like **compounding annually** (interest paid yearly) or **compounding monthly** (interest paid monthly)

- ✓ If $a\%$ p.a. (per annum) is compounded monthly, then $\frac{a}{12}\%$ will be paid each month.
- ✓ Fortunately, the compound interest formula accounts for this, eliminating the need for separate adjustments.

The formula for calculating compound interest is:

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

Where,

FV is the future value

PV is the present value

n is the number of years

k is the number of compounding periods per year

r % is the nominal annual rate of interest

This formula is **given in the formula booklet**, you do not have to remember it

Be careful with the *k* value,

Compounding **annually** means $k = 1$

Compounding **half-yearly** means $k = 2$

Compounding **quarterly** means $k = 4$

Compounding **monthly** means $k = 12$

Your GDC will have a finance solver app on it which you can use to find the future value.

EG.

How much Joe need to invest now, to get a maturing values of $20000$$ in 5 years time,given interest at 8% per annum compounded twice annually? Give your answer to nearest dollar?

$$20000 = A\cdot\left(1+\frac{0.08}{2}\right)^{2\cdot 5} \text{ (Calculator)}$$

$$A = \frac{20000}{\left(1+\frac{0.08}{2}\right)^{2\cdot 5}} \cong 13511$$

DEPRECIATION

Depreciation refers to the decrease in value of an asset as time passes.

The most common examples of depreciation are the value of cars and technology

If the depreciation happens at a constant rate then it is compound depreciation.

Here is the formula for calculating compound depreciation:

$$FV = PV \times \left(1-\frac{r}{100}\right)^{n}$$

Where,

FV is the future value

PV is the present value

n is the number of years

$r\%$ is the rate of depreciation

EG.

A truck was bought for $\$40000$ and depreciated by 8% each year.

a) Find the value of truck after 5 years?

b) By how much did it depreciate?

a) Value of the truck after 5 years;

$40000 \cdot (1 - 0{,}08)^5 = 26363$

b) $40000 - 26363 = 13637$

AMORTISATION

Amortisation is the process of repaying a loan over a fixed period of time

Typically, questions regarding amortization focus on mortgages (used for home purchases) or loans taken out for a large purchase

Interest will be paid on the original amount

- ✓ Each repayment that is made will partly repay the original loan and partly pay the interest on the loan
- ✓ As payments are made the amount owed will decrease and so the interest paid will decrease
- ✓ As you continue to repay a loan more of the repayment goes on the loan and less on the interest

Your GDC should be used to solve questions involving loans

- Use the **finance solver mode** (sometimes called the TVM (time value of money) solver)
- *N* will be the number of **repayment periods** (remember to include months and years if necessary)
- *I (%)* is the interest rate
- *PV* is the amount that was borrowed at the start – as this has been received it will be entered as a **positive** number
- *PMT* is the payments made per period – this is repaying the loan so will be a **negative** number
- *FV* is the future value (this will be zero as the loan will be paid off at the end of the period)
- *P/Y* is the number of payments per year, usually 12 as payments are made monthly
- *C/Y* is the **compounding periods** per year
- *PMT* is the time of the year or month the payment is made (assume this is the end unless told otherwise)

Leave the section that you need to find out blank and fill in all other sections. Your GDC will fill in the last part for you.

The total amount repaid will be **a little more** than the original loan plus *I* % of the original loan.

➡ You could use the following formula too:

$$R = \frac{i \times P}{1-(1+i)^{-n}}$$

$i =$ interest rate
$R =$ Periodic payment
P= amount owed
n= no. of periods

E.G. ➡ Find the annual repayment on a $\$300{,}000$ mortgage, over 20 years, with a 6% APR, compounded annually:

➡ Finding $R, w/i = 0.06, n = 20, P = 300000$: $R = \frac{0.06 \times 300000}{1-(1+0.06)^{-20}} = \frac{18000}{0.6882} = \$26155.19$

E.G. ➡ Let's solve the same problem with GDC.

N	$I\%$	PV	PMT	FV	P/Y	C/Y	PMT
20	6	300000	GDC will find for you	0	1	1	END

PMT= $-\ \$26155.19$

ANNUITIES

An annuity is a fixed sum of money paid to someone at specified intervals over a fixed period of time.

Most commonly this will be because of an initial lump sum investment which is then repaid at regular intervals with a predetermined interest rate.

Your GDC should be used to solve questions involving annuities

Use the **finance solver mode** (sometimes called the TVM (time value of money) solver)

- *N* will be the number of **payment periods** (remember to include months and years if necessary)
- *I (%)* is the interest rate
- *PV* is the amount that was invested – as this has been invested it will be entered as a **negative number**
- *PMT* is the amount paid per period – as this is being received it will be a **positive number**
- *FV* is the future value (for an annuity this will be **zero** as the balance at the end of the payment period will be zero)
- *P/Y* is the number of payments per year
- *C/Y* is the **compounding periods** per year
- *PMT* is the time of the year or month the payment is made (usually the start)

Leave the section that you need to find out blank and fill in all other sections. Your GDC will fill in the last part for you

The formula for computing an annuity is as follows:

$$FV = A\frac{(1+r)^n - 1}{r}$$

- Where

FV is the future value

A is the amount invested

n is the number of years

r% is the interest rate as a decimal (e.g. at 6%, r = 0.06)

This formula is actually an interpretation of geometric sequences sum formula.

✓ Try to remember the difference between amortization and annuities:

with **amortization** you are **paying** money out

with **annuities** you are **receiving** money

E.G. ➡ You invest $2000 at the end of each year, for 8 years, at a fixed interest rate of 5%. What will be the value at the end of the 8 years?

$$F.V. = 2000 + 2000(1.05) + 2000(1.05)^2 + \cdots ... + 2000(1.05)^7$$

$$FV = 2000 \times \frac{(1+0.05)^8 - 1}{0.05} = \$19098.22$$

Let's solve with GDC also,

N	$I\%$	PV	PMT	FV	P/Y	C/Y	PMT
8	5	0	-2000	GDC will find for you	1	1	END

FV=$19098.22

TECH TO SOLVE EQUATIONS

In this section, we will see how to solve specific challenging equations by GDC only.

- SYSTEMS You will have seen systems of 2 variables with 2 equations. With these, you must find an ' x ' & ' y ' that fits both simultaneously.
- MATRIX There are multiple ways, but you will need to be able to convert the equations to matrix
- Form with just the coefficients:

$$\begin{pmatrix} 2x+y-z=2 \\ x+3y+2z=1 \\ 2x+4y+6z=6 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & 1 & -1 & | & 2 \\ 1 & 3 & 2 & | & 1 \\ 2 & 4 & 6 & | & 6 \end{pmatrix}$$

EG. Solve.

$$\begin{aligned} 2x+y-z&=2 \\ x+3y+2z&=1 \\ 2x+4y+6z&=6 \end{aligned}$$

TI-84. ➡ [2 nd] ➡ $[x^{-1}]$ ➡ EDIT ➡ ENTER on 1: $[A]$ ➡ Change to 3×4 ➡ Type matrix

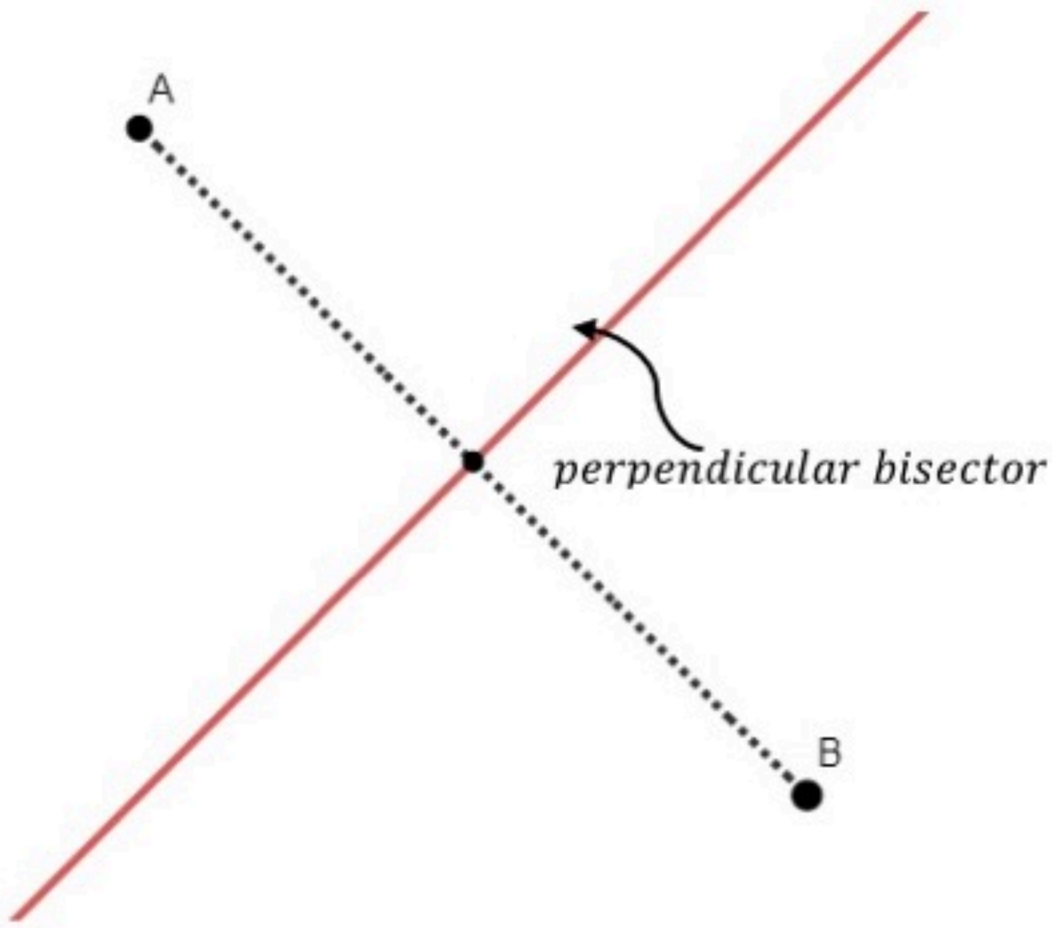
[2nd] ➡ x^{-1} ➡ MATH ➡ B: rref( ) ➡ [2nd] ➡ x^{-1} ➡ ENTER on 1: $[A]3\times4$ ➡ ENTER

➡ gives you $\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$ Answer: $\begin{aligned} x&=2 \\ y&=-1 \\ z&=1 \end{aligned}$

✓ For polynomials, you can use your GDC's poly root finder, or you can graph the equations and find the x-intercepts.

PERPENDICULAR BISECTORS

A bisector is a line that cuts halfway between two points.



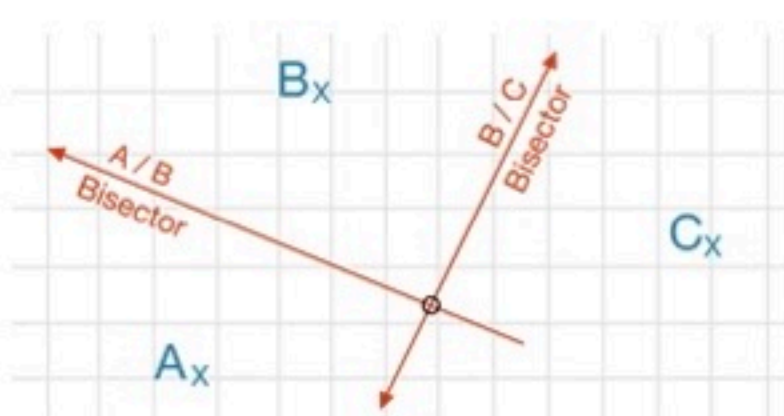
Properties

➡ These have certain properties:

Every point on the perpendicular bisector is equidistant from $A\&B$.

Every point left of the line is closer to A, and vice versa.

3-points ⟷ If you have 3 points, all you need is two (or three) perp. bisectors, and their intersection point is equidistant from $A, B\&C$, all at once:



EG.

Consider the points $R(2{,}8)$ and $S(0{,}12)$;
i) Find the midpoint of RS.
ii) Find the equation of perpendicular bisector of points R and S.

i) $m(1{,}10)$
ii) $m_{RS} = \frac{12-8}{0-2} = -2$ gradient of perpendicular bisector is $\frac{1}{2}$

$$y - 10 = \frac{1}{2}(x - 1)$$
$$2y - 20 = x - 1$$
$$x - 2y + 19 = 0$$

VORONOI DIAGRAMS

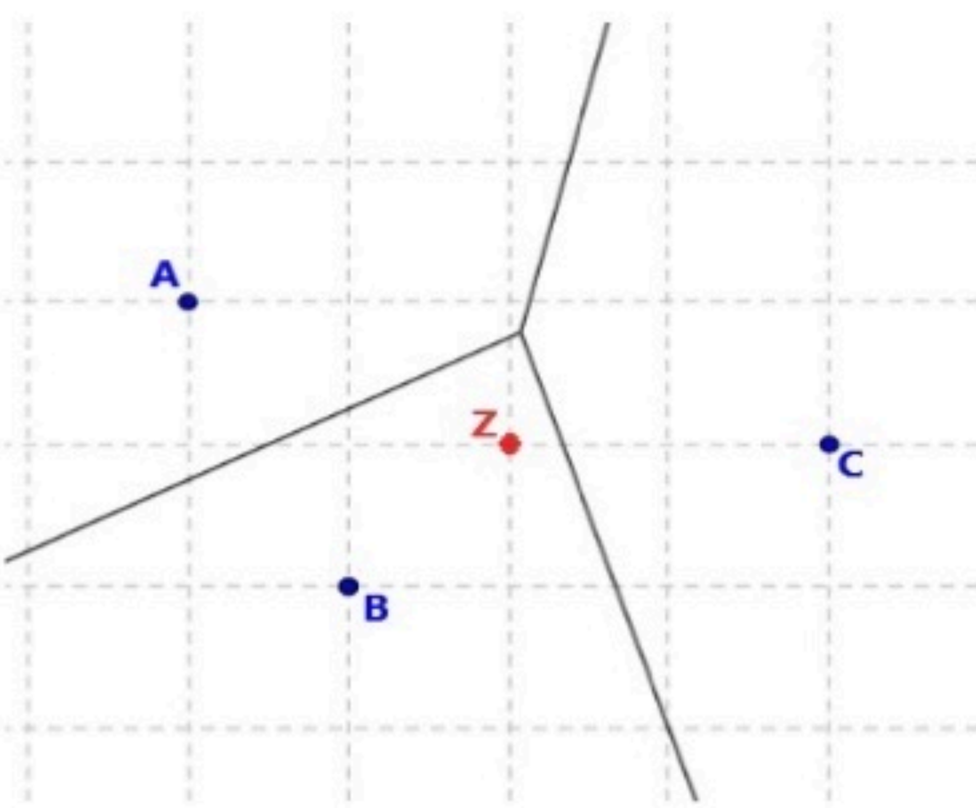
A **Voronoi diagram** shows the region containing the set of all points which are **closer** to one given **site** than to any other **site** on the diagram

A **site** is located at the coordinates of a specific place of interest on a Voronoi diagram

The **edges** of each region will be the **perpendicular bisector** of two of the sites

The **vertices** of each region are the **intersections** of **three** of these perpendicular bisectors

The perpendicular bisectors of three individual points will always intersect at the point that is **equidistant** from the three points



Z is the closest to B, because it's in the cell of the B

HOW ARE VORONOI DIAGRAMS DRAWN?

1. First, the perpendicular bisector of the line segment joining each pair of sites will be constructed.
2. These should be constructed using dashed lines as only a part of each line will be needed for the final diagram.
3. The points of intersection of these perpendicular bisectors will create the **vertices**.
4. Each perpendicular bisector should stop when it meets another perpendicular bisector.
5. Remove the part of the perpendicular bisector that is not in the region of the two sites.
6. No perpendicular bisector should cross over another. That will form the regions, or cells.

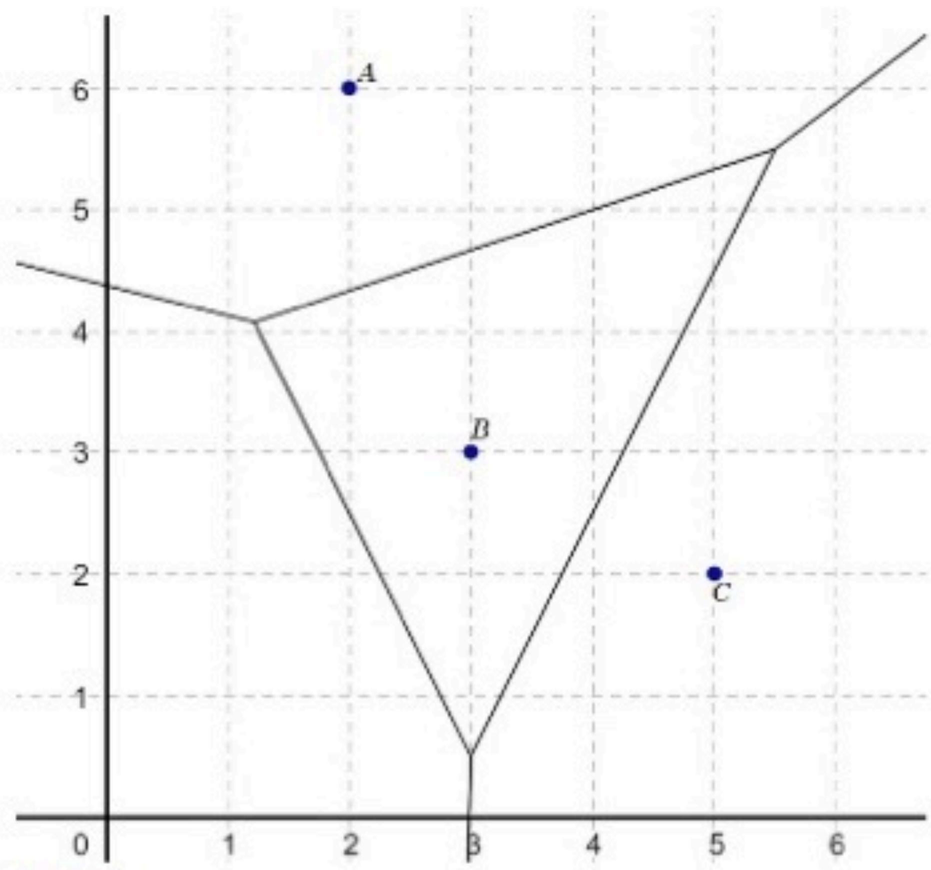
EG.

The points A, B, C are on the Voronoi diagram.

a) In empty region, point D is missing from the diagram. Determine the the coordinates of point D.

b) A point E is located at (5,5). Find the distance from E to nearest point.

c) Find the equation of perpendicular bisector between B and C.



a) Examine the diagram and understand that missing point is symmetric to given points.

point $D(1,2)$

b) Point $E(5.5)$ is located in the cell of point B so nearest point is B.

$$d = \sqrt{(5-3)^2 + (5-3)^2} = 2\sqrt{2}$$

c) We are searching for perpendicular bisector line between B and C.

$$B(3,3)\ C(5,2)$$

$$m = \frac{2-3}{5-3} = \frac{-1}{2} \qquad m_2 = 2$$

midpoint: $\frac{5+3}{2}, \frac{2+3}{2} \rightarrow \left(4, \frac{5}{2}\right)$

$$2 = \frac{y - \frac{5}{2}}{x-4} \qquad y = 2x - \frac{11}{2}$$

INFERENTIAL STATISTICS

On questions, there will be a claim for a characteristic of the population,

for example, the mean weight of students μ is 70 kg,

the proportion p of women is 45%,

the eating disorder is independent of the gender, etc.

We will state:

Null Hypothesis H_0 : the claim (affirmative)

Alternative hypothesis H_1 : the negation of the claim

We investigate a sample of the population against this characteristic.

Is the result close enough to the claim?

If NOT, we reject H_0

If YES, we do not have enough evidence to reject H_0

But what does it mean "close enough"?

This is determined by a so-called significance level a,

The significance level is usually	10% $a = 0.10$	5% $a = 0.05$	1% $a = 0.01$

This is in fact is the probability to reject the H_0 while it is true.

For example, $a = 0.05$ means: we reject results far away from the claim H_0, in a way that the probability to make a mistake is 5%.

P-VALUE

This is also crucial to all these tests. The p-value tells you:

If H_0 is true, then what is the probability of seeing this set of data, just by chance?

We compare the probability of the p-value with the significance level, and according to that, we will accept or reject our null hypothesis.

- If p-value < significance level: reject H_0
- If p-value > significance level: do not reject H_0.

T-TEST

Hypothesis test for two means μ_1 and μ_2 (t-test)

Question will give us some data. This data can be one-sample or two sample.

We test if they are close enough or not, to draw a conclusion for the population means μ_1 and μ_2. Assuming that the distribution of each population is normal, we perform the so-called t-test:

CLAIM: for the population means μ_1 and μ_2

$\mu_1 = \mu_2$ against	$\mu_1 \neq \mu_2$	$\mu_1 > \mu_2$ or $\mu_1 < \mu_2$
	2 -tailed test	1 -tailed test

Then, we state

[null hypothesis]	$H_0 : \mu_1 = \mu_2$
[alternative hypothesis]	$H_1 : \mu_1 \neq \mu_2$ or $\mu_1 > \mu_2$ or $\mu_1 < \mu_2$

We use GDC and GDC gives us the p-value.

Conclusion,

IF	THEN
p-value $< a$	we reject H_0

- ✓ If we are testing one sample against a whole population, this is a one-sample test.
- ✓ If we are just comparing two samples, this is a two-sample test.

TI-84 /→1-Sample: STAT → ⇆ TESTS → 2: t-TEST → Inpt: ↔Stats ENTER → Enter Values

2-Sample: STAT → ⇆ TESTS → 4:2-TEST → Inpt: ↔Stats ENTER → Enter Values

TI-Inspire/ 1-Sample: Menu → 6:Stats → 7:Stats tests → 2:t Tests → ↔ Stats → ...

2-Sample: Menu → 6:Stats → 7:Stats tests → 4:2-Sample t Test → ↔ Stats → ...

EG.

The lengths, in cm, of green beans in different plants are measured to find out whether there is any difference between them.

Plant 1	18	23	21	25	24	25	18	16	24
Plant 2	26	24	25	28	27	25	26	24	23

a) Write down the null and alternative hypotheses.
b) Find whether this a one-tailed test or a two-tailed test.
c) Find the t-value and p-value for a t-test at the 5% significance level.
d) Write down the conclusion to the test.

Sol: a) $H_0: \bar{x}_1 = \bar{x}_2$ (there is no difference between the lengths in Plant 1 and the lengths in the Plant 2)

$H_1: \bar{x}_1 \neq \bar{x}_2$ (there is a difference between the lengths in Plant 1 and the lengths in the Plant 2)

b) This is a two-tailed test because we are checking for any differences, therefore $H_1: \bar{x}_1 \neq \bar{x}_2$.

c) t-value $= -2.999$, p-value $= 0.011$ (using by GDC)

d) $0.011 < 0.05$,so we accept the alternative hypothesis: there is a significant difference between the two plants.

χ^2 TEST FOR GOODNESS OF FIT

They give us a list of observed frequencies,

$$f_1, f_2, \dots, f_n$$

CLAIM: they follow a given distribution

$$p_1, p_2, \dots, p_k$$

We state,

[null hypothesis]	H_0 : data follow the distribution
[alternative hypothesis]	H_1 : data do not follow the distribution

We use GDC,

We enter observed frequencies in List 1 and expected frequencies in List 2

List 1	List 2
f_1	Np_1
f_2	Np_2
...	...
f_n	Np_n

N = sum of frequencies in List 1
We also enter degrees of freedom $d.f. = n - 1$

GDC will give us p-value and x^2 $statistics$

Conclusion,

IF	THEN
p-value $< a$ (or χ^2 statistic $> \chi^2$ critical)	we reject H_0

★ χ^2 critical will be given if necessary.

$T1 - 84$ ➡ STAT ➡ EDIT ➡ obs. in L_1, Exp in L_2 ➡ [STAT] ➡ [TESTS] ➡ D:... ➡ L_1, L_2, DE, ENTER

EG.

There are 180 women living in a neighborhood. Data on number of the children they have is shown in the table.

Number of children	0	1	2	3	4	5
Frequency	32	51	47	28	15	7

a) Write down the table of expected values.
b) Write down the number of degrees of freedom.
c) The critical value is $\mathbf{15.534}$. Determine the results of a goodness of fit test at the 20% significance level to find out whether the data fits a uniform distribution. Remember to write down the null and alternative hypotheses.

Sol: a)

Number of children	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$
Frequency	30	30	30	30	30	30

b) $v = (6-1) = 5$

c) H_0 : The data satisfies a uniform distribution.

H_1 : The data does not satisfy a uniform distribution.

d) $x^2 = 49.73$ and ρ-value $= 1.5713$

$x^2 < 15.534$ or p-value > 0.2 so accept the null hypothesis. The data satisfies a uniform distribution.

χ^2 TEST FOR INDEPENDENCE

We are answering with this test: are these two variables independent of each other. ie.: If you are in a certain category of A, does it affect the probability of being in certain categories of B ?

Hypotheses:

You need to choose a null (H_0) hypothesis, and an alternate (H_1) hypothesis. H_0 is always that they are independent.

Expected frequencies:

Your GDC will calculate a probability of the variables being independent by comparing it to the table's expected values.

You can calculate each exp. value with this formula:

$$\left(\frac{\text{ROW TOTAL}}{\text{TOTAL}}\right) \times \left(\frac{\text{COL. TOTAL}}{\text{TOTAL}}\right) \times \text{TOTAL}$$

Degrees of freedom:

You will often be asked to calculate this, although it is another thing the GDC does automatically. But also:

$$\text{Degs. of Freedom} = (\text{rows} - 1) \times (\text{columns} - 1)$$

Final testing:

To perform the χ^2-best on your GDC, do the following:

TI-84/→ [2ND] [MATRIX] → EDIT → [ENTER] → Enter table →
[2ND] [QUIT] → [START] → ↔ TESTS → [C: x^2 - TESTS] …

TI-nspire /→ [MENU] → [7:MATRICES] → [1:…] → [1:…] → [ENTER] →
Enter data → [TAB] → [CTRL] [STO→] → Enter name →
[ENTER] → [MENU] → [6:…] → [7:…] → [8:…] → [ENTER] x2

Conclusion,

$$Reject\ H_0\ if\ p-value < Significant\ Level \quad OR \quad x^2 > Critical\ value$$

$$Do\ not\ Reject\ H_0\ if\ p-value > Significant\ Level \quad OR \quad x^2 < Critical\ value$$

EG.

Hailey was interested to find out whether preference for accessory was dependent on gender. She asked sixty people at work and the results are shown in the table.

Accessory	Sunglasses	Ring	Bracelet	Watch	Totals
Female	4	10	8	6	28
Male	10	5	5	12	32
Totals	14	15	13	18	60

a) Calculate the expected number of males who prefer rings.
b) Calculate the expected number of females who prefer bracelets.
c) Find the χ^2 value.

Sol:

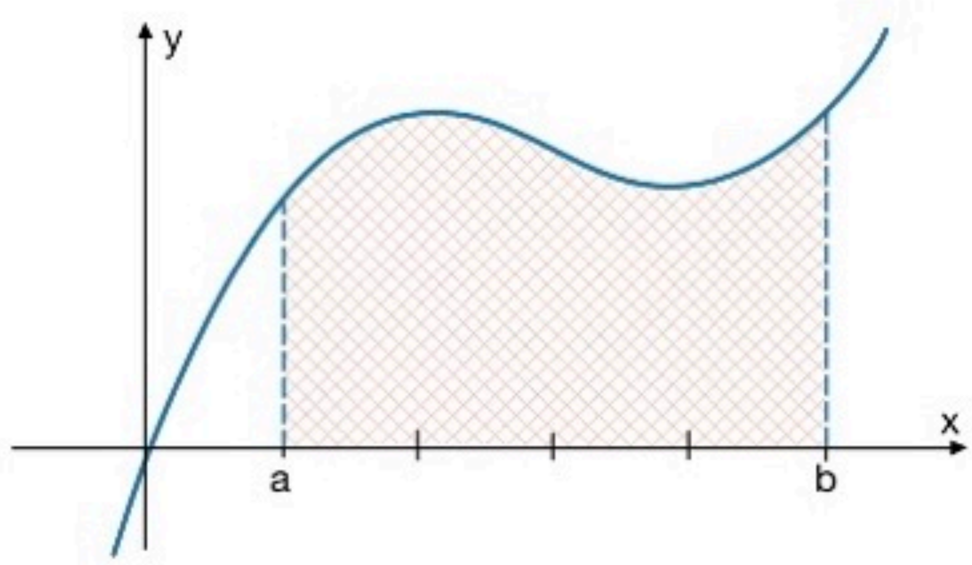
a) $60 \cdot \frac{32}{60} \cdot \frac{15}{60} = 8$

b) $60 \cdot \frac{28}{60} \cdot \frac{13}{60} = 6.1$

c) x^2 test statistic = 6.6935 (using technology)

TRAPEZOIDAL RULE

Consider the function $f(x)$ below and the area under the curve from $x = a$ to $x = b$.

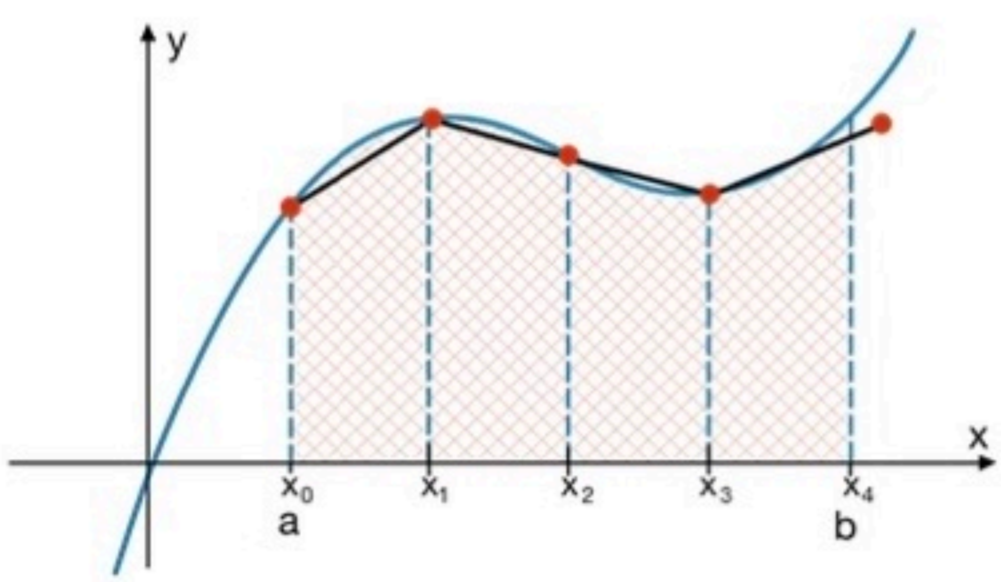


We will present an approximative way to estimate the area:
Let us divide the shaded region into four pieces of equal width

$$h = \frac{b-a}{4}$$

using the points $x_0 = a, x_1, x_2, x_3, x_4 = b$ (as shown below)

with y-coordinates y_0, y_1, y_2, y_3, y_4.



We will calculate the each trapezoid.

$$A_1 = \frac{1}{2}(y_0 + y_1)h, \quad A_2 = \frac{1}{2}(y_1 + y_2)h,$$
$$A_3 = \frac{1}{2}(y_2 + y_3)h, \quad A_4 = \frac{1}{2}(y_3 + y_4)h$$

Their sum is

$$A = \frac{1}{2}h(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + y_3 + y_4)$$
$$A = \frac{1}{2}h[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

This seems to be a good approximation of the shaded area. The more pieces we use the better approximation.
In general, if we divide the area into n pieces the formula takes the form of the so-called trapezoidal rule:

$$A = \int_a^b y\,dx \approx \frac{1}{2}h[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

Where,

$$h = \frac{b-a}{n}$$

EG.

Estimate the area under a curve over the interval $1 \le x \le 9$, with x - and y-values given in the following table.

x	1	3	5	7	9
y	6	14	11	4	5

Sol:

Need to find the area of each trapezoid and then sum these areas.

Area of trapezoid 1 $= \frac{1}{2}(6 + 14) \times 2 = 20$
Area of trapezoid 2 $= \frac{1}{2}(14 + 11) \times 2 = 25$
Area of trapezoid 3 $= \frac{1}{2}(11 + 4) \times 2 = 15$
Area of trapezoid 4 $= \frac{1}{2}(4 + 5) \times 2 = 9$
Area under the curve $= 20 + 25 + 15 + 9 = 69$